Lecture 8: Calculus and Differential Equations

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EE201: Computer Applications. See Textbook Chapter 9.

Numerical Methods

- MATLAB provides many functions that support numerical solutions to common math problems:
  - Integration and Differentiation (Calculus)
  - Finding zeros of a function
  - Solving ordinary differential equations
  - Many others
- Numerical analysis provides answers as numbers, not closed-form solutions as in analytical solutions (see *next* lecture for symbolic math in MATLAB).

The integral of *f(x)* is the area *A* under the curve of *f* (*x*) from *x* = *a* to *x* = *b*.

$$A = \int_a^b f(x)dx$$

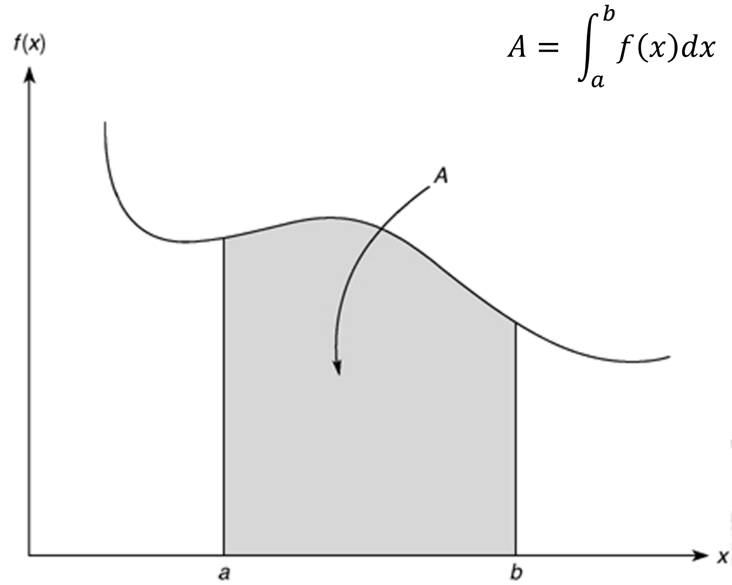
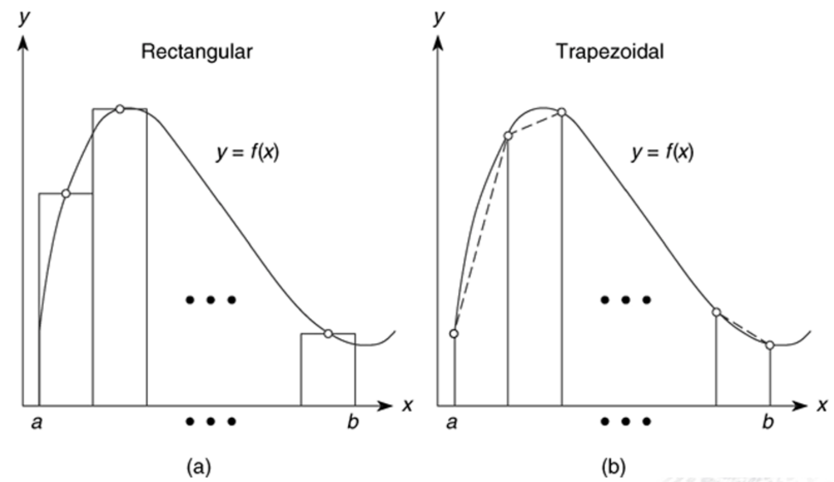


Illustration of Numerical Integration: (a) rectangular method and (b) more accurate trapezoidal method.



Example

$$A = \int_0^{\pi} \sin(x)\,dx = [-\cos(x)]_0^{\pi} = 1 - (-1) = 2$$

`trapz(x,y)`

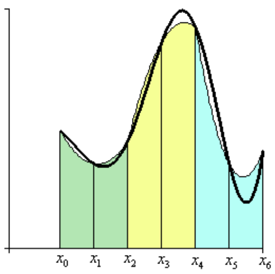
Uses trapezoidal integration to compute the integral of `y` with respect to `x`, where the array `y` contains the function values at the points contained in the array `x`.

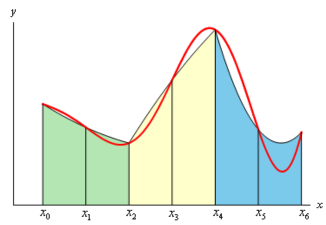
```
>> x = linspace(0,pi,10);
>> y = sin(x);
>> A = trapz(x,y)
A =
    1.9797

>> x = linspace(0,pi,100);
>> y = sin(x);
>> A = trapz(x,y)
A =
    1.9998
```


Simpson's Rule

- Another approach to numerical integration is Simpson's Rule, which divides the integration range [a, b] into an even number of sections and uses a different quadratic function to represent the integrand for each panel.





Important numerical integration functions:

`quad(fun, a, b)` `quad(fun, a, b, tol)`	Uses an adaptive Simpson's rule to compute the integral of the function whose handle is `fun`, with `a` the lower limit and `b` the upper limit. The function `fun` must accept a vector argument. The parameter `tol` is optional, and indicates the specified error tolerance.
`quadl(fun,a,b)`	Uses Lobatto quadrature to compute the integral of the function `fun`. The rest of the syntax is identical to `quad`.
`dblquad(fun, a, b, c, d)`	computes the integral of f(x,y) from x = a to b, and y = c to d. The function `fun` must accept a vector argument x and scalar y, and it must return a vector result.
`triplequad(fun,a,b,c,d,e,f)`	computes the integral of f(x,y,z) from x = a to b, y = c to d, and z = e to f. The function must accept a vector x, and scalar y and z.

Although the `quad` and `quadl` functions are more accurate than `trapz`, they are restricted to computing the integrals of functions and cannot be used when the integrand is specified by a set of points. For such cases, use the `trapz` function.

MATLAB function `quad` implements an adaptive version of Simpson's rule, while the `quadl` function is based on an adaptive Lobatto integration algorithm.

To compute the integral of sin(*x*) from 0 to π, type

```
>> A = quad(@sin,0,pi)
```

The answer given by MATLAB is 2.0000, which is correct. We use `quadl` the same way; namely,

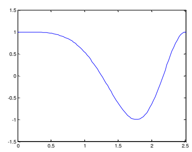
```
>>A = quadl(@sin,0,pi).
```


To integrate cos(x^2) from 0 to $\sqrt{2\pi}$, create the function in an m-file:

```
function yy = cossq(x)
yy = cos(x.^2);
```

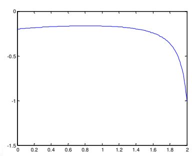
Note that we must use array exponentiation. Then `quad` function is called as follows:

```
>> quad(@cossq, 0, sqrt(2*pi))
ans =
    0.6119
```



Or you can use an anonymous function:

```
>> f = @(x)(1./(x.^3 - 2*x - 5));
>> quad(f, 0, 2)
ans =
    -0.4605
```



Double and Triple Integrals

`A = dblquad(fun, a, b, c, d)` computes the integral of *f(x,y)* from *x* = *a* to *b*, and *y* = *c* to *d*. Example: $f(x,y) = xy^2$.

```
>> fun = @(x,y) x.*y^2;
>> A = dblquad(fun, 1, 3, 0, 1)
A =
     1.3333
```

$$\int_c^d \int_a^b f(x,y)dxdy$$

`A = triplequad(fun, a, b, c, d, e, f)` computes the triple integral of *f(x,y, z)* from *x* = *a* to *b*, *y* = *c* to *d*, and *z* = *e* to *f*. Example: $f(x,y,z) = (xy - y^2)/z$.

```
>> fun = @(x,y,z)(x*y - y^2)/z;
>> A = triplequad(fun, 1,3, 0,2, 1,2)
A =
     1.8484
```

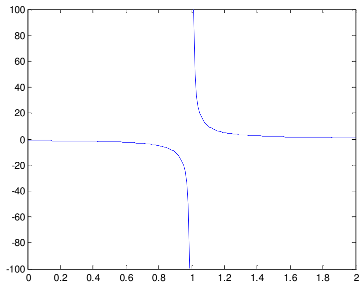
$$\int_e^f \int_c^d \int_a^b f(x,y,z)dxdydz$$

Note: The function must accept a vector *x*, but scalar *y* and *z*.

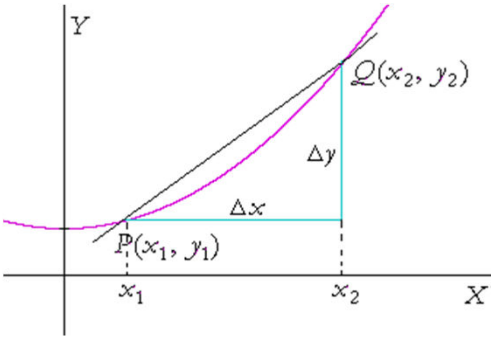
Be careful: function singularity

```
>> f = @(x) ( 1./(x-1));
>> quad(f, 0, 2)
Warning: Infinite or Not-
a-Number function value
encountered.
> In quad at 113
ans =
   NaN
```

$$\int_0^2 \frac{1}{1-x} dx$$



Numerical differentiation: Illustration of estimating the derivative *dy*/*dx*.



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} \approx \frac{y_2 - y_1}{x_2 - x_1}$$

MATLAB provides the `diff` function to use for computing derivative estimates.

`d = diff(y)`, where `y` is a vector of *n* elements, the result is a vector `d` containing *n* – 1 elements that are the differences between adjacent elements in `y`. That is:

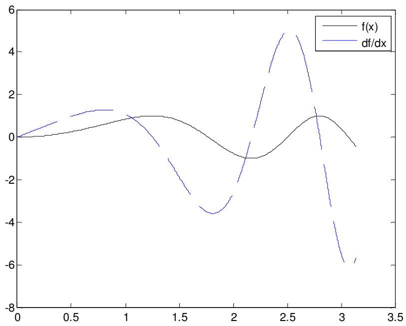
d=[*y*(2)-*y*(1), *y*(3)-*y*(2),..., *y*(*n*)-*y*(*n*-1)]

For example:

```
>> y = [5, 7, 12, -20];
>> diff(y)
ans =
      2      5    -32
```

Example

```
step = 0.001;
x = 0 : step : pi;
y = sin(x.^2);
d = diff(y)/step;
% an approximation
% to derivative
% 2.*x.*cos(x.^2)
plot(x,y,'k',x(2:end),d,'--');
legend('f(x)', 'df/dx');
```



Ordinary Differential Equations

- An ordinary differential equation (ODE) is an equation containing ordinary derivatives of the dependent variable.
- An equation containing partial derivatives with respect to two or more independent variables is a partial differential equation (PDE).
- We limit ourselves to ODE that must be solved for a given set of initial conditions.
- Solution methods for PDEs are an advanced topic, and we do not look at them.

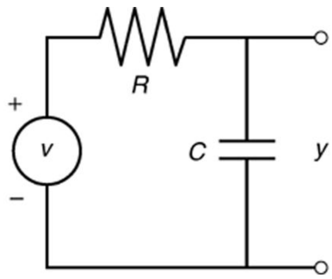
Several Methods

- Several numerical methods to solve ODEs.
- Examples include:
  - **Euler and Backward Euler methods**
  - Predictor-Corrector method
  - First-order exponential integrator method
  - **Runge-Kutta methods**
  - Adams-Moulton methods
  - Gauss-Radau methods
  - Adams-Bashforth methods
  - Hermite–Obreschkoff methods
  - Fehlberg methods
  - Parker–Sochacki methods
  - Nyström methods
  - Quantized State Systems methods

Multiple Solvers

- MATLAB offers multiple ODE solvers, each uses different methods.
- `Ode23`: Solves non-stiff differential equations, low order method.
- `ode45`: Solves non-stiff differential equations, medium order method: *uses a combination of fourth- and fifth-order Runge-Kutta methods*.
- `ode23s`: Solves stiff differential equations, low order method.
- `ode15i`: Solves fully implicit differential equations, variable order method.
- And so on.
- We will limit ourselves to the `ode45` solver.

Example: Find the response of the first-order RC circuit .



$$\tau \frac{dy}{dt} + y = 0$$

$$y(0) = V_c \quad (I.C.)$$

$$y(t) = y(0)e^{-t/\tau} \quad (natural\ response)$$

$$\tau \frac{dy}{dt} + y = V_s$$

$$y(0) = V_c \quad (I.C.)$$

$$y(t) = V_s + (y(0) - V_s)e^{-t/\tau} \quad (total\ response)$$

$$\dot{y}(t) = \frac{dy}{dt}$$

$$\ddot{y}(t) = \frac{d^2y}{dt^2}$$

Solving First-Order Differential Equations

First write the equation as *dy/dt = f(t,y)* then solve it using this syntax:

```
[t,y] = ode45(@f,tspan,y0)
```

where `@f` is the handle of the function file whose inputs must be *t* and *y*, and whose output must be a column vector representing *dy/dt*; that is, *f(t,y)*. The number of rows in the output column vector must equal the order of the equation.

The array `tspan` contains the starting and ending values of the independent variable *t*, and optionally any intermediate values.

The array `y0` contains the initial values of *y*. If the equation is first order, then `y0` is a scalar.

The circuit model for zero input voltage V_s and $\tau = 0.1$ is:

$$0.1 \times \frac{dy}{dt} + y = 0$$

And the i.c. is $y(0) = 2$ V.

First re-write the equation in the required format:

$$\frac{dy}{dt} = -10y$$

Next define the following function file. Note that the order of the input arguments must be *t* and *y*.

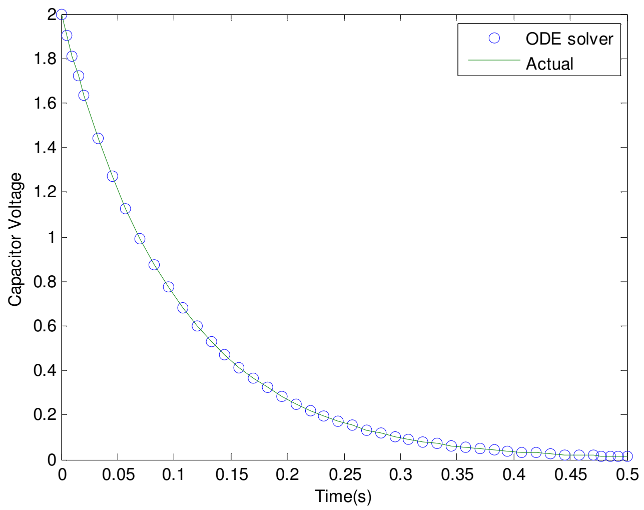
```
f = @(t,y) -10*y;
```


The solver is called as follows, and the solution plotted along with the analytical solution `y_true`. The initial condition is $y(0) = 2$.

```
f = @(t,y) -10*y;
[t, y] = ode45(f, [0 0.5], 2);
y_analytical = 2*exp(-10*t);
plot(t,y,'o', t, y_analytical);
legend('ODE solver', 'Actual');
xlabel('Time(s)');
ylabel('Capacitor Voltage');
```

Note that we need not generate the array `t` to evaluate `y_analytical`, because `t` is generated by the `ode45` function.
The plot is shown on the next slide.

Free (natural) response of an RC circuit (decaying exponential).



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The circuit model for input voltage $V_s = 10V$ and $\tau = 0.1$:

$$0.1 \times \frac{dy}{dt} + y = 10$$

And the i.c. is $y(0) = 2$ V.

First re-write the equation in the required format:

$$\frac{dy}{dt} = -10y + 100$$

Next define the following function file. Note that the order of the input arguments must be *t* and *y*.

```
f = @(t,y) -10*y+100;
```

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The solver is called as follows, and the solution plotted along with the analytical solution `y_true`. The initial condition is $y(0) = 2$.

```
f = @(t,y) -10*y+100;
[t, y] = ode45(f, [0 0.5], 2);
y_analytical = 10+(2-10)*exp(-10*t);
plot(t,y,'o', t, y_analytical);
legend('ODE solver', 'Actual');
xlabel('Time(s)');
ylabel('Capacitor Voltage');
```

Note that we need not generate the array `t` to evaluate `y_analytical`, because `t` is generated by the `ode45` function.
The plot is shown on the next slide.

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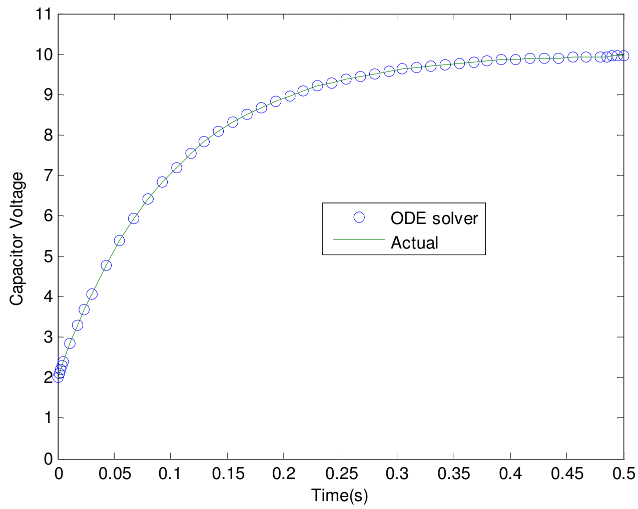
Natural plus forced (total) response of an RC circuit (increasing exponential).



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The circuit model for input voltage $V_s = 10e^{-t/0.3} \sin\left(\frac{2\pi t}{0.03}\right)$ and $\tau = 0.1$:

$$0.1 \times \frac{dy}{dt} + y = 10e^{-t/0.3} \sin\left(\frac{2\pi t}{0.03}\right)$$

And assume the i.c. is $y(0) = 0$ V.

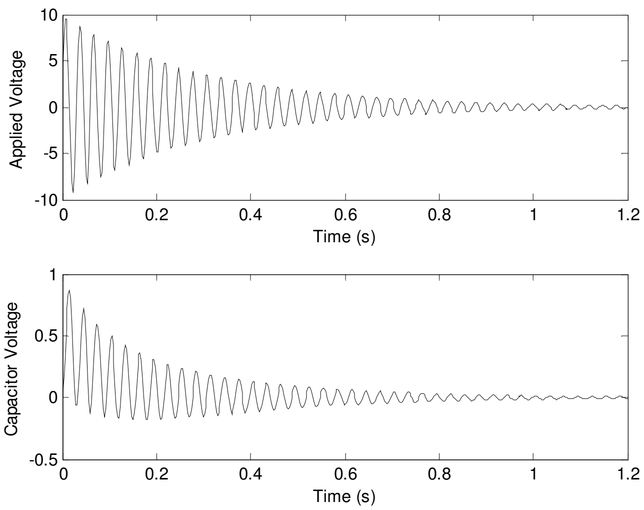
First re-write the equation in the required format:

$$\frac{dy}{dt} = -10y + 100e^{-t/0.3} \sin\left(\frac{2\pi t}{0.03}\right)$$

Next define the following function file. Note that the order of the input arguments must be *t* and *y*.

```
f = @(t,y) -10*y+100* ...
exp(-1*t/0.3).*sin(2*pi*t/0.03);
```


Result



Extension to Higher-Order Equations

To use the ODE solvers to solve an equation of 2nd order or higher, you must first write the equation as a set of first-order equations.

Example:

$$5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 4y = f(t)$$

By re-arranging to get the highest derivative:

$$\frac{d^2y}{dt^2} = \frac{1}{5}f(t) - \frac{4}{5}y - \frac{7}{5}\frac{dy}{dt}$$

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Example (*Continue*)

$$\frac{d^2y}{dt^2} = \frac{1}{5}f(t) - \frac{4}{5}y - \frac{7}{5}\frac{dy}{dt}$$

We then change variables: $x_2 = dy/dt$
Hence: $dx_2/dt = d^2y/dt^2$
Also: $x_1 = y$. Hence we have two equations:

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2$$

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Example (*Continue*)

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2$$

This form is sometimes called the Cauchy form or the state-variable form.

We now define a function that accepts two values of x and then computes the values of dx_1/dt and dx_2/dt and stores them in a column vector.

Example (Code)

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{1}{5}\sin(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2$$

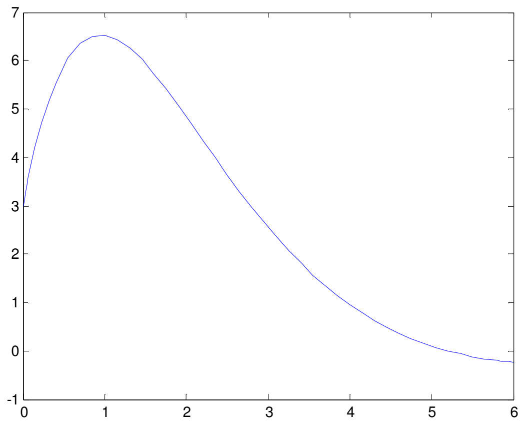
```
d = @(t,x) [x(2); sin(t)/5-4*x(1)/5-7*x(2)/5];
[t, x] = ode45(d, [0 6], [3 9]);
```

Here $x(0) = 3$ and $\dot{x}(0) = 9$, and we solve for $0 \le t \le 6$. Also $f(t) = \sin(t)$.

Note `x` is a matrix with two columns. The first column contains the values of x_1 at the various times generated by the solver; the second column contains the values of x_2.

If you type `plot(t, x)`, you will obtain a plot of both x_1 and x_2 versus `t`. Thus, type `plot(t, x(:,1))` to see the result for `y`.

Result



HW: Alternative Solution

Define the function in an m-file:

```
function xdot = d(t, x)
xdot(1) = x(2);
xdot(2) = (1/5)*(sin(t)-4*x(1)-7*x(2));
xdot = [xdot(1); xdot(2)];
```

Use the function to solve the ODE:

```
[t, x] = ode45(@d, [0 6], [3 9]);
% notice the need to use handles
plot(t, x(:,1));
```

Homework

- Solve as many problems from Chapter 9 as you can
- Suggested problems:
- Solve: 9.1, 9.4, 9.14, 9.16, 9.23, 9.27, 9.31, 9.34.

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