EE 221: Signal Analysis & Systems
Prof. Mohammed Hawa

Fourier Series Equations: Practice Problems

Q1. Consider the following periodic signals:

- **(i)** $x(t) = \sin(4t) + \cos(8t) + 7 + \cos(16t)$
- **(ii)** $x(t) = \cos^2(t)$
- **(iii)** $x(t) = \cos(t) + \sin(2t) + \cos(3t - \pi/3)$

(a) Find the trigonometric Fourier series coefficients for each of the above signals.
(b) Find the compact Fourier series coefficients for each of the above signals.
(c) Find the complex exponential Fourier series coefficients for each of the above signals.

Q1. Solution.
(i) Using trigonometric Fourier series we can write the periodic $x(t)$ as an infinite sum of sinusoids $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$. In other words, $x(t) = \frac{a_0}{2} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \cdots + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \cdots$. Since the given signal is already written in this form (sum of sinusoids), we can read the coefficients simply *by inspection* (there is no need to perform any integration). All we need is the fundamental frequency of $x(t)$, which is $\omega_0 = 4$ rad/s, since the fundamental frequency of $x(t)$ is the greatest common divisor of the frequencies of the added sinusoids. Hence,

$$\begin{aligned} x(t) &= \sin(4t) + \cos(8t) + 7 + \cos(16t) \\ &= 7 + \sin(\omega_0 t) + \cos(2\omega_0 t) + \cos(4\omega_0 t) \end{aligned}$$

And the trigonometric Fourier series coefficients are:
$a_0/2 = 7$,
$a_2 = 1$,
$a_4 = 1$,
otherwise $a_n = 0$
$b_1 = 1$,
otherwise $b_n = 0$

Similarly, in compact Fourier series, $x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \frac{c_0}{2} + c_1 \cos(\omega_0 t - \theta_1) + c_2 \cos(2\omega_0 t - \theta_2) + c_3 \cos(3\omega_0 t - \theta_3) + \cdots$. Hence, by inspection and noticing that $\sin(\omega_0 t) = \cos(\omega_0 t - \pi/2)$, the compact Fourier series coefficients are:

$c_0/2 = 7,$
$c_1 = 1, \theta_1 = \pi/2,$
$c_2 = 1, \theta_2 = 0,$
$c_4 = 1, \theta_4 = 0,$
otherwise $c_n = 0, \theta_n = 0$

In complex exponential Fourier series, we have $x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} = \alpha_0 e^{j0} + \alpha_1 \, e^{j\omega_0 t} + \alpha_{-1} \, e^{-j\omega_0 t} + \alpha_2 \, e^{j2\omega_0 t} + \alpha_{-2} \, e^{-j2\omega_0 t} + \cdots$. Hence,

$$x(t) = 7 + \sin(\omega_0 t) + \cos(2\omega_0 t) + \cos(4\omega_0 t)$$

$$x(t) = 7e^{j0} + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + \frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} + \frac{e^{j4\omega_0 t} + e^{-j4\omega_0 t}}{2}$$

And the complex exponential Fourier series coefficients are:

$\alpha_0 = 7,$
$\alpha_1 = -0.5j,$
$\alpha_{-1} = 0.5j,$
$\alpha_2 = 0.5,$
$\alpha_{-2} = 0.5,$
$\alpha_4 = 0.5,$
$\alpha_{-4} = 0.5,$
otherwise $\alpha_n = 0$

(ii) Using the well-known trigonometric identity

$$x(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t) = \frac{1}{2} + \frac{1}{2}\cos(\omega_0 t)$$

The fundamental frequency is $\omega_0 = 2$ rad/s. Also, using Euler's identity,

$$x(t) = \frac{1}{2} + \frac{1}{2}\cos(\omega_0 t) = \frac{1}{2}e^{j0} + \frac{1}{2} \times \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Trigonometric Fourier series coefficients:
$a_0/2 = 1/2,$
$a_1 = 1/2,$
otherwise $a_n = 0$
all $b_n = 0$

Compact Fourier series coefficients:
$c_0/2 = 1/2,$
$c_1 = 1/2, \theta_1 = 0,$
otherwise $c_n = 0, \theta_n = 0$

Complex Fourier series coefficients:
$\alpha_0 = 1/2,$
$\alpha_1 = 0.25,$
$\alpha_{-1} = 0.25,$
otherwise $\alpha_n = 0$

(iii) The fundamental frequency of $x(t)$ is $\omega_0 = 1$ rad/s (the greatest common divisor (GCD))

$$\begin{aligned} x(t) = \cos(t) + \sin(2t) + \cos\left(3t - \frac{\pi}{3}\right) \\ &= \cos(\omega_0 t) + \sin(2\omega_0 t) + \cos\left(3\omega_0 t - \frac{\pi}{3}\right) \\ &= \cos(\omega_0 t) + \sin(2\omega_0 t) + [A\cos(3\omega_0 t) + B\sin(3\omega_0 t)] \end{aligned}$$

where,

$$A = 1 \times \cos\left(\frac{\pi}{3}\right) = 1/2$$
$$B = 1 \times \sin\left(\frac{\pi}{3}\right) = \sqrt{3}/2$$

Trigonometric Fourier series coefficients:
$a_0/2 = 0,$
$a_1 = 1,$
$a_3 = 1/2,$
otherwise $a_n = 0$
$b_2 = 1,$
$b_3 = \sqrt{3}/2,$
otherwise $b_n = 0$

Compact Fourier series coefficients
(notice, for example, that $\sin(2\omega_0 t) = \cos(2\omega_0 t - \pi/2)$):
$c_0/2 \ = 0,$
$c_1 = 1, \theta_1 = 0,$
$c_2 = 1, \theta_2 = \pi/2,$
$c_3 = 1, \theta_3 = \pi/3,$
otherwise $c_n = 0, \theta_n = 0$

$$x(t) = \cos(\omega_0 t) + \sin(2\omega_0 t) + \cos\left(3\omega_0 t - \frac{\pi}{3}\right)$$
$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} + \frac{e^{j\left(3\omega_0 t - \frac{\pi}{3}\right)} + e^{-j\left(3\omega_0 t - \frac{\pi}{3}\right)}}{2}$$

Complex Fourier series coefficients:
$\alpha_0 \ = 0,$
$\alpha_1 = 0.5,$
$\alpha_{-1} = 0.5,$
$\alpha_2 = -0.5j,$
$\alpha_{-2} = 0.5j,$
$\alpha_3 = 0.5e^{-j\pi/3},$
$\alpha_{-3} = 0.5e^{j\pi/3},$
otherwise $\alpha_n = 0$

Q2. If the two halves of one period of a periodic signal are identical in shape except that one is the negative of the other, the periodic signal is said to have *half-wave symmetry*. In other words, if a periodic signal $x(t)$ with period T_0 satisfies half-wave symmetry, then

$$x\left(t - \frac{T_0}{2}\right) = x(-t)$$

In this case, show that all the *even*-numbered harmonics vanish and that the *odd*-numbered harmonic trigonometric coefficients are given by

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos(n\omega_0 t)\, dt$$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin(n\omega_0 t)\, dt$$

Q2. Solution.

$$a_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{0} x(t)\cos(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

Changing variables $\zeta = t + \frac{T_0}{2}$ or $t = \zeta - \frac{T_0}{2}$ for the first integral

$$a_n = \frac{2}{T_0}\int_{\zeta=0}^{T_0/2} x\left(\zeta - \frac{T_0}{2}\right)\cos\left(n\omega_0\left(\zeta - \frac{T_0}{2}\right)\right)dt + \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{0}^{T_0/2} -x(\zeta)\cos(n\omega_0\zeta - n\pi)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{0}^{T_0/2} -x(\zeta)(-1)^n\cos(n\omega_0\zeta)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

Since

$$\cos(n\omega_0 t - n\pi) = \cos(n\pi)\cos(n\omega_0 t) = (-1)^n\cos(n\omega_0 t)$$

For n even

$$a_n = \frac{2}{T_0}\int_{0}^{T_0/2} -x(\zeta)(1)\cos(n\omega_0\zeta)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$
$$= 0$$

For n odd

$$a_n = \frac{2}{T_0}\int_{0}^{T_0/2} -x(\zeta)(-1)\cos(n\omega_0\zeta)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$
$$= \frac{4}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)\,dt$$

A similar derivation can be performed for b_n to get:

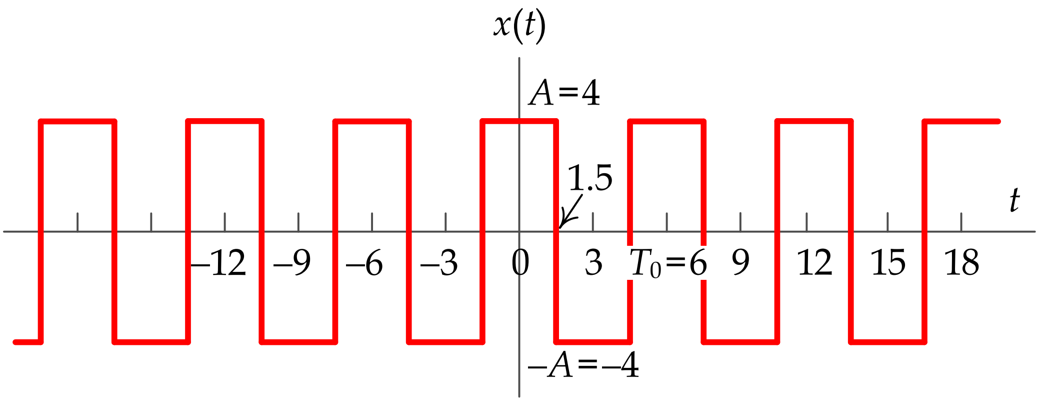
For n even

$$b_n = 0$$

For n odd

$$b_n = \frac{4}{T_0}\int_{0}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt$$

Q3. Consider the following periodic signal:



(a) Write a mathematical expression for this signal.
(b) Find the exponential Fourier series coefficients for this signal.
(c) Find the trigonometric Fourier series coefficients for this signal.
(d) Find the compact Fourier series coefficients for this signal.

Q3. Solution.

$$x(t) = \text{rep}_6\left\{8 \,\text{rect}\left(\frac{t}{3}\right)\right\} - 4$$

$$\alpha_0 = 0$$

$$\alpha_n = \frac{8 \times 3}{6} \text{sinc}\left(\frac{n \times 3}{6}\right) = 4 \,\text{sinc}\left(\frac{n}{2}\right), \quad n \neq 0$$

$$a_0 = 0$$

$$a_n = \frac{2 \times 8 \times 3}{6} \text{sinc}\left(\frac{n \times 3}{6}\right) = 8 \,\text{sinc}\left(\frac{n}{2}\right)$$

$$b_n = 0$$

$$c_n = \left|\frac{2 \times 8 \times 3}{6} \text{sinc}\left(\frac{n \times 3}{6}\right)\right| = \left|8 \,\text{sinc}\left(\frac{n}{2}\right)\right|$$

$$\theta_n = 0° \text{ or } 180°$$