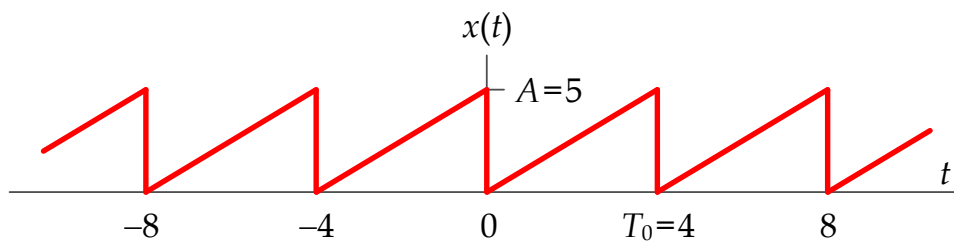


Fourier Series: Practice Problems

Q1. For the following signal $x(t) = \text{rep}_{T_0} \left\{ \frac{A}{2} \text{saw} \left(\frac{(t-T_0/2)}{T_0} \right) \right\} + \frac{A}{2}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q1. Answer.

$$\alpha_0 = \frac{A}{2}$$

$$\alpha_n = j \frac{A}{2n\pi}, \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \left| \frac{A}{2n\pi} \right| \quad \angle \alpha_n = -\theta_n = \begin{cases} \pi/2, & n > 0 \\ 0, & n = 0 \\ -\pi/2, & n < 0 \end{cases}$$

$$a_0 = A$$

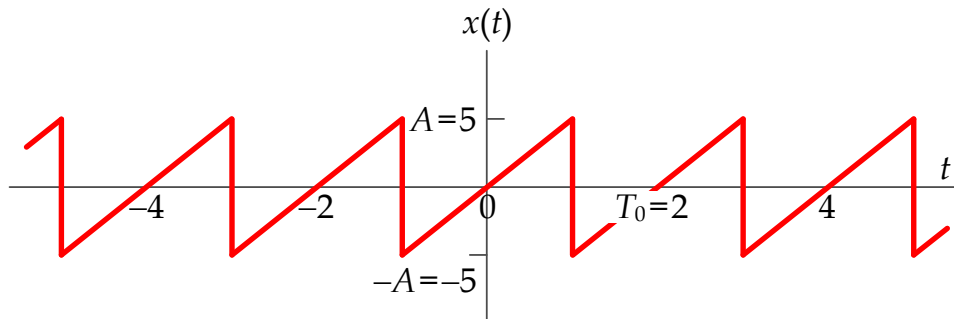
$$a_n = 0, \quad n \neq 0$$

$$b_n = \frac{-A}{n\pi}$$

To see a step-by-step solution of this problem, please watch the YouTube video (https://www.youtube.com/watch?v=_s_P5XsaKdY). Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0 .

To see a step-by-step solution of a similar problem to this one, please watch the YouTube video (https://www.youtube.com/watch?v=yVF_POXZBlw). Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0 .

Q2. For the following signal $x(t) = \text{rep}_{T_0} \left\{ A \text{ saw} \left(\frac{t}{T_0} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q2. Answer.

$$\alpha_0 = 0$$

$$\alpha_n = j \frac{A}{n\pi} \cos(n\pi) = j \frac{A}{n\pi} (-1)^n, \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \left| \frac{A}{n\pi} \right| \quad \angle \alpha_n = -\theta_n = \begin{cases} \pi/2, & n \text{ even} \\ 0, & n = 0 \\ -\pi/2, & n \text{ odd} \end{cases}$$

$$a_0 = 0$$

$$a_n = 0, \quad n \neq 0$$

$$b_n = \frac{2A}{n\pi} (-1)^{n+1}$$

To see a step-by-step solution of a similar problem to this one, please watch the YouTube video

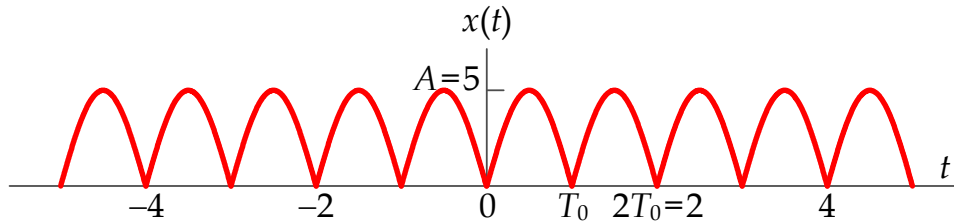
(<https://www.youtube.com/watch?v=XZf49AZ35bY>)

and also the YouTube video

(<https://www.youtube.com/watch?v=0NzyqNTRObI>).

Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0 , and that the video's C_n corresponds to the complex exponential Fourier series coefficients α_n .

Q3. For the following signal $x(t) = \left| A \sin\left(\frac{\omega_0}{2} t\right) \right|$ (i.e., full-wave rectified signal), determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q3. Solution.

Notice that the fundamental frequency is

$$\omega_0 = \frac{2\pi}{T_0}$$

But the fundamental period T_0 is only half cycle of the original sine wave, rather than its full cycle.

$$\alpha_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_n = \frac{1}{T_0} \int_0^{T_0} A \sin\left(\frac{\omega_0}{2} t\right) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_n = \frac{A}{T_0} \int_0^{T_0} \left[\frac{e^{\frac{j\omega_0 t}{2}} - e^{-\frac{j\omega_0 t}{2}}}{2j} \right] e^{-jn\omega_0 t} dt$$

$$\alpha_n = \frac{A}{2jT_0} \int_0^{T_0} e^{\frac{j\omega_0 t}{2}} e^{-jn\omega_0 t} dt - \frac{A}{2jT_0} \int_0^{T_0} e^{-\frac{j\omega_0 t}{2}} e^{-jn\omega_0 t} dt$$

$$\alpha_n = \frac{A}{2jT_0} \int_0^{T_0} e^{j(\frac{1}{2}-n)\omega_0 t} dt - \frac{A}{2jT_0} \int_0^{T_0} e^{-j(\frac{1}{2}+n)\omega_0 t} dt$$

$$\alpha_n = \frac{A}{2jT_0} \left[\frac{1}{j\left(\frac{1}{2} - n\right)\omega_0} e^{j\left(\frac{1}{2} - n\right)\omega_0 t} \right]_0^{T_0} - \frac{A}{2jT_0} \left[\frac{1}{-j\left(\frac{1}{2} + n\right)\omega_0} e^{-j\left(\frac{1}{2} + n\right)\omega_0 t} \right]_0^{T_0}$$

$$\alpha_n = \frac{A}{2jT_0} \left[\frac{e^{j\left(\frac{1}{2} - n\right)\omega_0 T_0} - e^0}{j\left(\frac{1}{2} - n\right)\omega_0} \right] - \frac{A}{2jT_0} \left[\frac{e^{-j\left(\frac{1}{2} + n\right)\omega_0 T_0} - e^0}{-j\left(\frac{1}{2} + n\right)\omega_0} \right]$$

$$\alpha_n = \frac{-A}{2T_0} \left[\frac{e^{j\left(\frac{1}{2} - n\right)2\pi} - 1}{\left(\frac{1}{2} - n\right)\omega_0} \right] - \frac{A}{2T_0} \left[\frac{e^{-j\left(\frac{1}{2} + n\right)2\pi} - 1}{\left(\frac{1}{2} + n\right)\omega_0} \right]$$

But notice that,

$$e^{j\left(\frac{1}{2} - n\right)2\pi} = e^{j(1 - 2n)\pi} = \cos((1 - 2n)\pi) + j \sin((1 - 2n)\pi) = -1 + j0, \quad \forall n$$

$$e^{j\left(\frac{1}{2} + n\right)2\pi} = e^{j(1 + 2n)\pi} = \cos((1 + 2n)\pi) + j \sin((1 + 2n)\pi) = -1 + j0, \quad \forall n$$

Hence,

$$\alpha_n = \frac{-A}{2T_0} \left[\frac{-2}{\left(\frac{1}{2} - n\right)\omega_0} \right] - \frac{A}{2T_0} \left[\frac{-2}{\left(\frac{1}{2} + n\right)\omega_0} \right]$$

$$\alpha_n = \frac{A}{T_0} \left[\frac{1}{\left(\frac{1}{2} - n\right)\left(\frac{2\pi}{T_0}\right)} \right] + \frac{A}{T_0} \left[\frac{1}{\left(\frac{1}{2} + n\right)\left(\frac{2\pi}{T_0}\right)} \right]$$

$$\alpha_n = A \left[\frac{1}{(1 - 2n)\pi} \right] + A \left[\frac{1}{(1 + 2n)\pi} \right]$$

$$\alpha_n = \frac{A}{\pi} \left[\frac{(1 + 2n) + (1 - 2n)}{(1 - 2n)(1 + 2n)} \right] = \frac{A}{\pi} \left[\frac{2}{1 - 2n + 2n - 4n^2} \right]$$

$$\alpha_n = \frac{2A}{\pi(1 - 4n^2)}, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_0 = \frac{2A}{\pi}$$

$$\alpha_n = \frac{2A}{\pi(1-4n^2)}, \quad n \neq 0$$

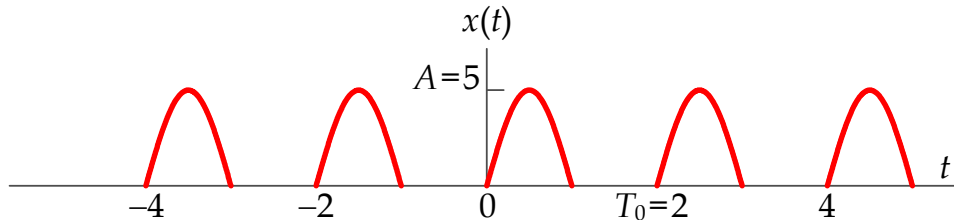
$$|\alpha_n| = \frac{c_n}{2} = \left| \frac{2A}{\pi(1-4n^2)} \right| \quad \angle \alpha_n = -\theta_n = \pi$$

$$a_0 = \frac{4A}{\pi}$$

$$a_n = \frac{4A}{\pi(1-4n^2)}, \quad n \neq 0$$

$$b_n = 0$$

Q4. For the following signal $x(t) = \sin(\omega_0 t) \text{rep}_{T_0} \left\{ \text{rect} \left(\frac{(t-T_0/4)}{T_0/2} \right) \right\}$ (i.e., half-wave rectified signal), determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q4. Answer.

$$\alpha_0 = \frac{A}{\pi}$$

$$\alpha_n = \begin{cases} 0 - j\frac{A}{4}, & n = 1 \\ \frac{A}{\pi(1-n^2)} - j0, & n = 2, 4, 6, 8, \dots \\ 0 - j0, & n = 3, 5, 7, 9, \dots \end{cases}$$

$$|\alpha_n| = \frac{c_n}{2} \qquad \angle \alpha_n = -\theta_n$$

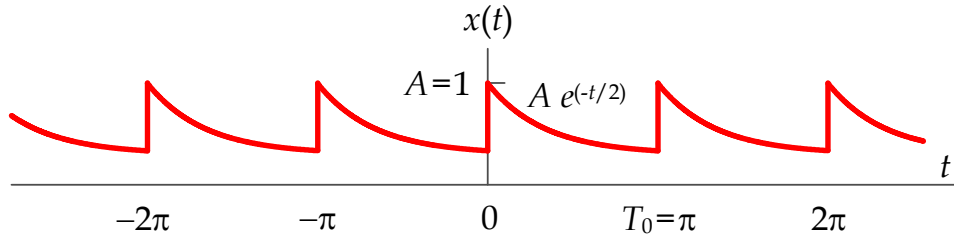
$$a_0 = \frac{2A}{\pi}$$

$$a_n = \begin{cases} \frac{2A}{\pi(1-n^2)}, & n = 2, 4, 6, 8, \dots \\ 0, & n = 1, 3, 5, 7, \dots \end{cases}$$

$$b_n = \begin{cases} \frac{A}{2}, & n = 1 \\ 0, & n = 2, 3, 4, 5, \dots \end{cases}$$

To see a step-by-step solution of this problem, please watch the YouTube video (<https://www.youtube.com/watch?v=SM20fAvKwgY>). Be careful though, and notice the difference between our definition of a_0 and the video's definition of a_0 .

Q5. For the following signal $x(t) = \text{rep}_{T_0} \left\{ A e^{(-t/2)} \text{rect} \left(\frac{t-T_0/2}{T_0} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q5. Solution.

$$\alpha_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_n = \frac{1}{T_0} \int_0^{T_0} A e^{-t/2} e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_n = \frac{A}{T_0} \int_0^{T_0} e^{-(jn\omega_0 + \frac{1}{2})t} dt$$

$$\alpha_n = \frac{A}{T_0} \left[\frac{1}{-(jn\omega_0 + \frac{1}{2})} e^{-(jn\omega_0 + \frac{1}{2})t} \right]_0^{T_0}$$

$$\alpha_n = \frac{A}{T_0} \times \frac{1}{-(jn\omega_0 + \frac{1}{2})} \left[e^{-(jn\omega_0 + \frac{1}{2})T_0} - e^0 \right]$$

$$\alpha_n = \frac{A}{j2\pi n + \frac{\pi}{2}} \left[1 - e^{-(j2\pi n + \frac{\pi}{2})} \right]$$

For n integer, the $2\pi n$ added to phase does not change the complex number

$$\alpha_n = \frac{A}{j2\pi n + \frac{\pi}{2}} \left[1 - e^{-\frac{\pi}{2}} e^{-j2\pi n} \right] = \frac{A}{j2\pi n + \frac{\pi}{2}} \left[1 - e^{-\frac{\pi}{2}} e^0 \right] = \frac{A[1 - e^{-\pi/2}]}{j2\pi n + \frac{\pi}{2}}$$

Now we perform complex number division with the help of the conjugate

$$\alpha_n = \frac{A[1 - e^{-\pi/2}]}{\frac{\pi}{2} + j2\pi n} \times \frac{\frac{\pi}{2} - j2\pi n}{\frac{\pi}{2} - j2\pi n} = \frac{A\frac{\pi}{2}[1 - e^{-\pi/2}] - jA(2\pi n)[1 - e^{-\pi/2}]}{\left(\frac{\pi}{2}\right)^2 + (2\pi n)^2}$$

$$\alpha_n = \frac{A\frac{\pi}{2}[1 - e^{-\pi/2}] - jA(2\pi n)[1 - e^{-\pi/2}]}{\left(\frac{\pi}{2}\right)^2 + (2\pi n)^2} = \frac{a_n}{2} - j\frac{b_n}{2}$$

So,

$$a_n = \frac{2A\frac{\pi}{2}[1 - e^{-\pi/2}]}{\left(\frac{\pi}{2}\right)^2 + (2\pi n)^2} = \frac{A}{\pi} \times \frac{1}{\left(\frac{1}{2}\right)^2 + (2n)^2} \times [1 - e^{-\pi/2}]$$

$$a_n = A \times \frac{2(1 - e^{-\pi/2})}{\pi} \times \frac{2}{1 + 16n^2} = A \times 0.5043 \times \frac{2}{1 + 16n^2}$$

And,

$$b_n = \frac{2A(2\pi n)[1 - e^{-\pi/2}]}{\left(\frac{\pi}{2}\right)^2 + (2\pi n)^2} = \frac{A}{\pi} \times \frac{4n[1 - e^{-\pi/2}]}{\left(\frac{1}{2}\right)^2 + (2n)^2}$$

$$b_n = A \times \frac{2(1 - e^{-\pi/2})}{\pi} \times \frac{8n}{1 + 16n^2} = 0.5043 A \left(\frac{8n}{1 + 16n^2} \right)$$

Hence,

$$a_0 = 1.0086 A$$

$$a_n = 0.5043 A \left(\frac{2}{1 + 16n^2} \right), \quad n \neq 0$$

$$b_n = 0.5043 A \left(\frac{8n}{1 + 16n^2} \right)$$

And for the complex exponential form,

$$\alpha_0 = 0.504 A$$

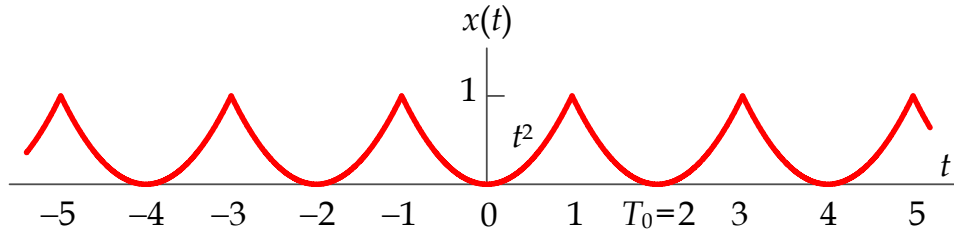
$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = 0.5043 A \left(\frac{1}{1 + 16n^2} \right) - j 0.5043 A \left(\frac{4n}{1 + 16n^2} \right), \quad n \neq 0$$

$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = 0.5043 A \left(\frac{1}{1 + 16n^2} \right) - j 0.5043 A \left(\frac{4n}{1 + 16n^2} \right), \quad n \neq 0$$

$$\begin{aligned} |\alpha_n| &= \frac{c_n}{2} = \sqrt{\frac{0.5043^2 A^2 \times 1}{(1 + 16n^2)^2} + \frac{0.5043^2 A^2 \times 16n^2}{(1 + 16n^2)^2}} \\ &= 0.5043 A \left(\frac{1}{\sqrt{1 + 16n^2}} \right) \end{aligned}$$

$$\begin{aligned} \angle \alpha_n &= -\theta_n = \tan^{-1} \left(\frac{-b_n}{a_n} \right) = \tan^{-1} \left(\frac{-0.5043 A \left(\frac{8n}{1 + 16n^2} \right)}{0.5043 A \left(\frac{2}{1 + 16n^2} \right)} \right) \\ &= \tan^{-1}(-4n) = -\tan^{-1}(4n) \end{aligned}$$

Q6. For the following signal $x(t) = t^2 \text{rep}_2 \left\{ \text{rect} \left(\frac{t}{2} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q6. Answer.

$$\alpha_0 = 1/3$$

$$\alpha_n = \frac{2 \times (-1)^n}{(n\pi)^2}, \quad n \neq 0$$

$$|\alpha_n| = \frac{c_n}{2} = \left| \frac{2 \times (-1)^n}{(n\pi)^2} \right| = \frac{2}{(n\pi)^2} \quad \angle \alpha_n = -\theta_n = \begin{cases} \pi, & n \text{ even} \\ 0, & n = 0 \\ -\pi, & n \text{ odd} \end{cases}$$

$$a_0 = 2/3$$

$$a_n = \frac{4 \times (-1)^n}{(n\pi)^2}, \quad n \neq 0$$

$$b_n = 0$$