# Signal Analysis Student Guide

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This handout provides you with a quick review of the main equations you studied for Fourier series, Fourier transform and other identities.

## 1 Fourier Series

The idea behind Fourier series is that you can express any *periodic* signal $x(t)$ as the sum of an infinite number of sinusoidal (cosine/sine) functions. There are three ways of doing that: complex form, trigonometric form and compact form.

### 1.1 Complex exponential Fourier series:

Any *periodic* signal $x(t)$ with period $T$ can be expanded into complex *exponential* Fourier series as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \cdot e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

where,

$$\alpha_n = \frac{1}{T} \int_{t_1}^{t_1+T} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

For real-valued $x(t)$ we have $\alpha_{-n} = \alpha_n^*$ (i.e., $|\alpha_{-n}| = |\alpha_n|$, $\angle\alpha_{-n} = -\angle\alpha_n$), $n = 0,\ 1\ ,\ 2,\ \ldots$

### 1.2 Trigonometric Fourier series:

Any *periodic* signal $x(t)$ with period $T$ can be expanded into *trigonometric* Fourier series as follows:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)), \quad \omega_0 = \frac{2\pi}{T}$$

where,

$$a_n = 2Re\{\alpha_n\} = \frac{2}{T} \int_{t_1}^{t_1+T} x(t) \cos(n\omega_0 t)\, dt, \quad n = 0, 1, 2, \ldots$$

$$b_n = -2Im\{\alpha_n\} = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin(n\omega_0 t)\,dt, \quad n = 0,\,1,\,2,\,\ldots$$

Notice that,

$$\alpha_n = \frac{1}{2}\left(a_n - jb_n\right), \quad n = 0,\,1,\,2,\,\ldots$$

### 1.3 Compact Fourier series:

Any *periodic* signal $x(t)$ with period $T$ can be expanded into *compact* Fourier series as follows:

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n), \quad \omega_0 = \frac{2\pi}{T}$$

where,

$$c_n = \sqrt{a_n^2 + b_n^2} = 2\left|\alpha_n\right|, \quad n = 0,\,1,\,2,\,\ldots$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{-Im\left\{\alpha_n\right\}}{Re\left\{\alpha_n\right\}}\right) = -\angle\alpha_n, \quad n = 0,\ 1\ ,\ 2,\ \ldots$$

Notice that,

$$a_n = c_n\cos\left(\theta_n\right) = 2Re\left\{\alpha_n\right\}, \quad n = 0,\ 1\ ,\ 2,\ \ldots$$

$$b_n = c_n\sin\left(\theta_n\right) = -2Im\left\{\alpha_n\right\}, \quad n = 0,\ 1\ ,\ 2,\ \ldots$$

Not only do you need to memorize the above equations, you also need to be able to use them to evaluate the Fourier series coefficients for any periodic function $x(t)$. To practice your skills, Table 1 below shows some periodic signals $x(t)$ with the corresponding Fourier series coefficients. Make sure you can obtain them yourself. The shape of the periodic signals $x(t)$ in the table will be shown later.

The **complex** Fourier series coefficients $\alpha_n$'s represent the **Fourier spectrum** of the signal $x(t)$, or simply, its **spectrum**. Since these coefficients are complex numbers, they generate two spectra: the **magnitude spectrum** and the **phase spectrum** of $x(t)$. The magnitude spectrum can also be drawn using the compact coefficients $c_n$ since $c_n = 2\left|\alpha_n\right|$. The $c_n$ spectrum, however, is called the **one-sided magnitude spectrum** (because $n \geq 0$), while the $\alpha_n$ spectrum is called the **two-sided magnitude spectrum** (because $-\infty < n < +\infty$).

| $x(t)$ | Fourier Series Coefficients |
|---|---|
| $rep_T\{A\ rect\,(t/\tau)\}$ | $\alpha_n = \frac{A\tau}{T}\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right),\ \lvert\alpha_n\rvert = \frac{A\tau}{T}\left\lvert\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right)\right\rvert,\ \angle\alpha_n = 0^\circ\ or\ 180^\circ$<br>$a_n = \frac{2A\tau}{T}\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right),\ b_n = 0$<br>$c_n = \frac{2A\tau}{T}\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right),\ \theta_n = 0^\circ$<br>Bandwidth: $B_{x(t)} = 1/\tau,$<br>DC: $\overline{x(t)} = A\tau/T$<br>Average Power: $P_x = \overline{x^2(t)} = A^2\tau/T$ |
| $rep_T\{2A\ rect\,(2t/T)\} - A$ | $\alpha_0 = 0,\ \alpha_n = A\ \text{sinc}\left(\frac{n}{2}\right),$<br>$\lvert\alpha_0\rvert = 0,\ \lvert\alpha_n\rvert = A\left\lvert\text{sinc}\left(\frac{n}{2}\right)\right\rvert,\ \angle\alpha_n = 0^\circ\ or\ 180^\circ$<br>$a_0 = 0,\ b_0 = 0,\ a_n = 2A\ \text{sinc}\left(\frac{n}{2}\right),\ b_n = 0$<br>$c_0 = 0,\ c_n = 2A\ \text{sinc}\left(\frac{n}{2}\right),\ \theta_n = 0^\circ$<br>Bandwidth: $B_{x(t)} = 2/T,$<br>DC: $\overline{x(t)} = 0$<br>Average Power: $P_x = \overline{x^2(t)} = 2A^2 - A^2 = A^2$ |
| $rep_T\{A\ \Delta\,(t/\tau)\}$ | $\alpha_n = \frac{A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \lvert\alpha_n\rvert = \frac{A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \angle\alpha_n = 0^\circ$<br>$a_n = \frac{2A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ b_n = 0$<br>$c_n = \frac{2A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \theta_n = 0^\circ$<br>Bandwidth: $B_{x(t)} = 1/\tau,$<br>DC: $\overline{x(t)} = A\tau/T$<br>Average Power: $P_x = \overline{x^2(t)} = 2A^2\tau/3T$ |
| $rep_T\{2A\ \Delta\,(2t/T)\} - A$ | $\alpha_0 = 0,\ \alpha_n = A\ \text{sinc}^2\left(\frac{n}{2}\right),$<br>$\lvert\alpha_0\rvert = 0,\ \lvert\alpha_n\rvert = A\ \text{sinc}^2\left(\frac{n}{2}\right),\ \angle\alpha_n = 0^\circ$<br>$a_0 = 0,\ b_0 = 0,\ a_n = 2A\ \text{sinc}^2\left(\frac{n}{2}\right),\ b_n = 0$<br>$c_0 = 0,\ c_n = 2A\ \text{sinc}^2\left(\frac{n}{2}\right),\ \theta_n = 0^\circ$<br>Bandwidth: $B_{x(t)} = 2/T,$<br>DC: $\overline{x(t)} = 0$<br>Average Power: $P_x = \overline{x^2(t)} = 4A^2/3 - A^2 = A^2/3$ |
| $rep_T\{A\ saw\,(t/\tau)\}$ | $\alpha_0 = 0,\ \alpha_n = \frac{-j2A}{n\omega_o T}\left[\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right],$<br>$\lvert\alpha_0\rvert = 0,\ \lvert\alpha_n\rvert = \frac{2A}{n\omega_o T}\left\lvert\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right\rvert,\ \angle\alpha_n = \pm 90^\circ$<br>$a_0 = 0,\ b_0 = 0,\ a_n = 0,\ b_n = \frac{4A}{n\omega_o T}\left[\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right]$<br>$c_0 = 0,\ c_n = \frac{-4A}{n\omega_o T}\left[\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right],\ \theta_n = -90^\circ$<br>Bandwidth: $B_{x(t)} = 3/\tau,$<br>DC: $\overline{x(t)} = 0$<br>Average Power: $P_x = \overline{x^2(t)} = A^2\tau/3T$ |
| $rep_T\{A\ saw\,(t/T)\}$ | $\alpha_0 = 0,\ \alpha_n = \frac{jA\cos(n\pi)}{n\pi},$<br>$\lvert\alpha_0\rvert = 0,\ \lvert\alpha_n\rvert = \left\lvert\frac{A\cos(n\pi)}{n\pi}\right\rvert,\ \angle\alpha_n = \pm 90^\circ$<br>$a_0 = 0,\ b_0 = 0,\ a_n = 0,\ b_n = \frac{-2A\cos(n\pi)}{n\pi}$<br>$c_0 = 0,\ c_n = \frac{2A\cos(n\pi)}{n\pi},\ \theta_n = -90^\circ$<br>Bandwidth: $B_{x(t)} = 3/T,$<br>DC: $\overline{x(t)} = 0$<br>Average Power: $P_x = \overline{x^2(t)} = A^2/3$ |

** $\omega_0 = \frac{2\pi}{T}$

# 2 Fourier Transform

The Fourier transform is a mathematical tool that converts a signal $x(t)$ from *time domain* into *frequency domain* as $\mathrm{X}(\omega)$. The inverse Fourier transform does the opposite.

## 2.1 Definition:

The Fourier transform of a general signal $x(t)$, whether *periodic* or *aperiodic*, is given by:

$$\mathrm{X}(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

and the inverse Fourier transform is:

$$x(t) = \mathscr{F}^{-1}\{\mathrm{X}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{X}(\omega)e^{j\omega t}d\omega$$

The Fourier transform $\mathrm{X}(\omega)$ represents the **Fourier spectrum density** of the signal $x(t)$. Notice that the Fourier spectrum density of *periodic* signals consist of a group of impulses $\delta(\omega)$'s, while the Fourier spectrum density of *aperiodic* signals is a smooth continuous curve. This is due to the fact that periodic signals are actually the sum of an infinite number of sinusoidals.

## 2.2 Properties of Fourier Transforms:

Usually to save time when evaluating Fourier transforms we avoid using the original integral and rely on using tables instead. See Table 2 (Selected Fourier Transform Pairs) and Table 3 (Properties of Fourier Transform). Make sure you memorize both tables.

| $x(t)$ | $\mathrm{X}(\omega) = \mathscr{F}\{x(t)\}$ |
|---|---|
| $\cos(\omega_0 t)$ | $\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$ |
| $\sin(\omega_0 t)$ | $-j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)$ |
| $e^{\pm j\omega_0 t}$ | $2\pi\delta(\omega \mp \omega_0)$ |
| $\mathrm{rect}(t)$ | $\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$ |
| $\mathrm{rect}\left(\frac{t}{\tau}\right)$ | $\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$ |
| $\Delta(t)$ | $\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$ |
| $\Delta\left(\frac{t}{\tau}\right)$ | $\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$ |
| $\mathrm{sinc}(t)$ | $\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$ |
| $\mathrm{sinc}\left(\frac{t}{2\pi}\right)$ | $2\pi\ \mathrm{rect}(\omega)$ |
| $saw\left(\frac{t}{\tau}\right)$ | $\frac{-2j}{\omega}\left[\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right) - \cos\left(\frac{\omega\tau}{2}\right)\right]$ |
| $\delta(t)$, direct delta function | 1 |
| 1 | $2\pi\delta(\omega)$ |
| $\mathrm{rep}_T\{\delta(t)\} = \sum_{n=-\infty}^{\infty}\delta(t-nT)$ | $\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{T}\right) = \omega_0\,\mathrm{rep}_{\omega_0}\{\delta(\omega)\}$ |
| $rep_T\{p(t)\}$ | $\sum_{n=-\infty}^{\infty} 2\pi\alpha_n\delta(\omega-n\omega_o)$ |
| $rep_T\left\{A\ rect\left(\frac{t}{\tau}\right)\right\}$ | $\sum_{n=-\infty}^{\infty} 2\pi\frac{A\tau}{T}\mathrm{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right)\ \delta(\omega-n\omega_o)$ |
| $rep_T\left\{A\ \triangle\left(\frac{t}{\tau}\right)\right\}$ | $\sum_{n=-\infty}^{\infty} 2\pi\frac{A\tau}{T}\mathrm{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right)\ \delta(\omega-n\omega_o)$ |
| $rep_T\left\{A\ saw\left(\frac{t}{T}\right)\right\}$ | $\sum_{n=-\infty}^{\infty} j2\pi A\frac{\cos(n\pi)}{n\pi}\delta(\omega-n\omega_o)$ |
| $u(t)$, unit step function | $\pi\delta(\omega) + \frac{1}{j\omega}$ |
| $sgn(t) = u(t) - u(-t)$ | $\frac{2}{j\omega}$ |
| $e^{-t^2/2}$ | $\sqrt{2\pi}\ e^{-\omega^2/2}$ |

Notice that all Fourier transforms in Table 2 are given in terms of angular frequency $\omega$ (rad/s) instead of frequency $f$ (Hertz). This is the convention we will use in this class. There is a factor of $2\pi$ that you have to be aware of between the Fourier transform $\mathrm{X}(\omega)$ and $\mathrm{X}(f)$. For example, $\mathscr{F}\{x(t) = \cos(\omega_0 t)\} = X(\omega) = \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$ but $\mathscr{F}\{x(t) = \cos(\omega_0 t)\} = X(f) = \frac{1}{2}\,\delta(f - f_0) + \frac{1}{2}\,\delta(f + f_0)$. Also $\mathscr{F}\{x(t) = \mathrm{rect}\,(t)\} = X(\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$ but $\mathscr{F}\{x(t) = \mathrm{rect}\,(t)\} = X(f) = \mathrm{sinc}\,(f)$. This difference is due to the fact that $\omega = 2\pi f$.

| **Property** | $x(t)$ | $\mathrm{X}(\omega) = \mathscr{F}\{x(t)\}$ |
|---|---|---|
| Linearity (superposition) | $ax(t) + by(t)$ | $a\mathrm{X}(\omega) + b\mathrm{Y}(\omega)$ |
| Complex conjugate | $x(t)^*$ | $\mathrm{X}^*(-\omega)$ |
| Symmetry | $x_{even}(t)$<br>$x_{odd}(t)$ | $\mathrm{X}_{even}(\omega)$, real<br>$\mathrm{X}_{odd}(\omega)$, imaginary |
| Duality | $\mathrm{X}(t)$ | $2\pi x(-\omega)$ |
| Reciprocal spreading | $x\left(\frac{t}{\tau}\right)$ | $\|\tau\|\,\mathrm{X}(\tau\omega)$ |
| Time shift (delay) | $x(t \pm \tau)$ | $\mathrm{X}(\omega)\,e^{\pm j\omega\tau}$ |
| Frequency shift | $x(t)\,e^{\pm j\omega_0 t}$ | $\mathrm{X}(\omega \mp \omega_0)$ |
| Modulation | $x(t)\cos\omega_0 t = \frac{x(t)}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$ | $\frac{1}{2}\left(\mathrm{X}(\omega - \omega_0) + \mathrm{X}(\omega + \omega_0)\right)$ |
| Differentiation | $\frac{dx(t)}{dt}$ | $j\omega\mathrm{X}(\omega)$ |
| Integration | $\int_{-\infty}^{t} x(\tau)d\tau$ | $\frac{1}{j\omega}\mathrm{X}(\omega) + \pi\mathrm{X}(0)\,\delta(\omega)$ |
| Time convolution | $x(t) \circledast y(t)$ | $\mathrm{X}(\omega)\mathrm{Y}(\omega)$ |
| Frequency convolution | $x(t) \cdot y(t)$ | $\frac{1}{2\pi}\left(\mathrm{X}(\omega) \circledast \mathrm{Y}(\omega)\right)$ |
| Reversal | $x(-t)$ | $\mathrm{X}(-\omega)$ |

# 3 Energy and Power Spectral Densities

## 3.1 Energy Spectral Density:

The **energy spectral density** (ESD) of a general signal $x(t)$ is defined as follows:

$$\mathrm{ESD} = \Psi_x(\omega) = |\mathrm{X}(\omega)|^2$$

The ESD is a function that describes the relative amount of energy of a given signal versus frequency. The total area under the ESD is the total energy in the signal $x(t)$ defined as $E_x$.

## 3.2 Parseval's Theorem:

The total energy in a general signal $x(t)$ can be calculated using two different methods as follows:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathrm{X}(\omega)|^2\,d\omega$$

### 3.3 Power Spectral Density:

The **power spectral density** (PSD) of a general signal $x(t)$ is defined as follows:

$$\text{PSD} = \mathrm{S}_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \left|\mathrm{X}_T(\omega)\right|^2$$

The PSD is a function that describes the relative amount of power of a given signal versus frequency. The total area under the PSD is the total power in the signal $x(t)$ defined as $P_x$.

Notice that the PSD of *periodic* functions consist of a group of impulses $\delta(t)$'s, while the PSD of *aperiodic* functions is a smooth continuous curve. This is due to the fact that periodic functions are actually the sum of an infinite number of sinusoidal functions.

Most of the signals we consider in communications theory exist for a long time, i.e., they are power signals. Some of them are periodic and some are aperiodic. Power signals have a PSD not an ESD (their ESD is infinite).

### 3.4 Average Power:

The average power in a general signal $x(t)$ can be calculated using two different methods as follows:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|x(t)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{S}_x(\omega) d\omega$$

## 4 Average and RMS

Consider a continuous message signal $x(t)$ and a discrete version of the message signal $x_n$ with a sampling period $\Delta t$ (and the sampling frequency $f_s = 1/\Delta t$):

### 4.1 Average of a message signal:

The average of the signal $x(t)$ is:

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt$$

and for the sampled version of $x(t)$, where we have $N$ samples $\triangle t$ apart, the average is:

$$\bar{x} = \frac{1}{N\cdot\Delta t} \sum_n x_n \cdot \Delta t = \frac{1}{N} \sum_n x_n, \quad \Delta t = \frac{1}{f_s}$$

### 4.2 RMS of a message signal:

The *root mean square* (rms) of the signal $x(t)$ is:

$$x_{rms} = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt}$$

and for the sampled version of $x(t)$, where we have $N$ samples $\triangle t$ apart, the rms is:

$$x_{rms} = \sqrt{\frac{1}{N \cdot \Delta t} \sum_n x_n^2 \cdot \Delta t} = \sqrt{\frac{1}{N} \sum_n x_n^2}, \quad \Delta t = \frac{1}{f_s}$$

### 4.3 Average Power of a message signal:

As we explained in Section 3.4, the *average power* in the signal $x(t)$ is given by:

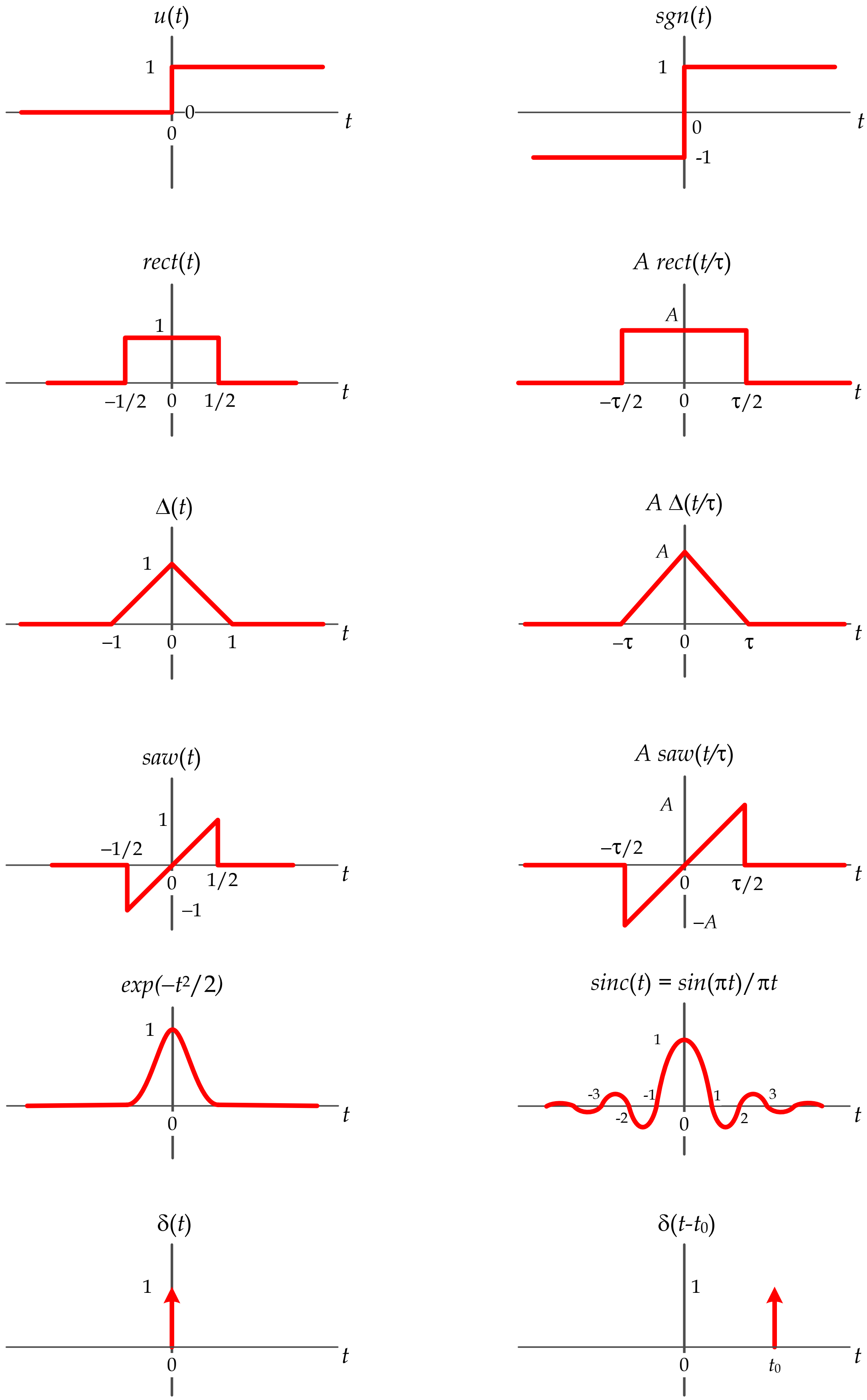
$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

and for the sampled version the average power is:

$$P_x = \frac{1}{N} \sum_n x_n^2$$

Notice that the rms value squared is actually the average power in the signal. This is because we are assuming a normalized load impedance of 1 Ω. Hence, the average power in $x(t)$ is:

$$P_x = x_{rms}^2$$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τ/2
–A
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exp(–t2/2)
1
0
t
sinc(t) = sin(πt)/πt
1
-3
-1
1
3
-2
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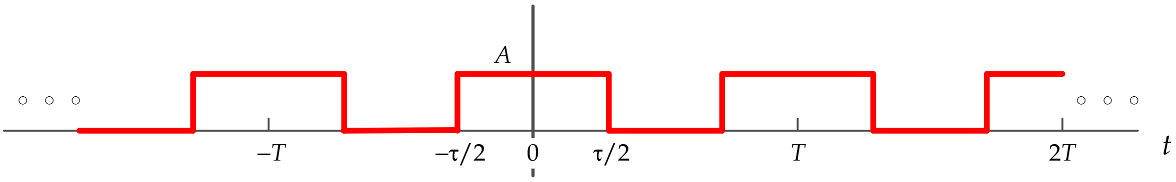
$rep_T\{A\ rect\ (t/\tau)\}$

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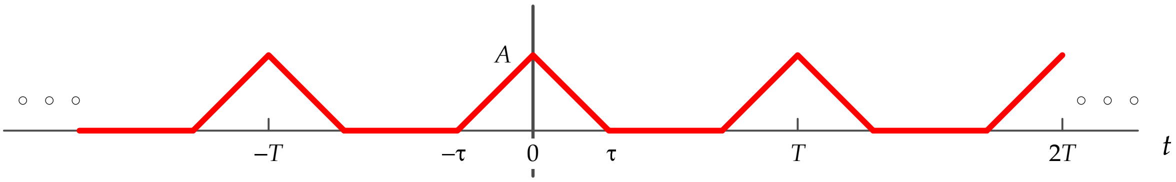
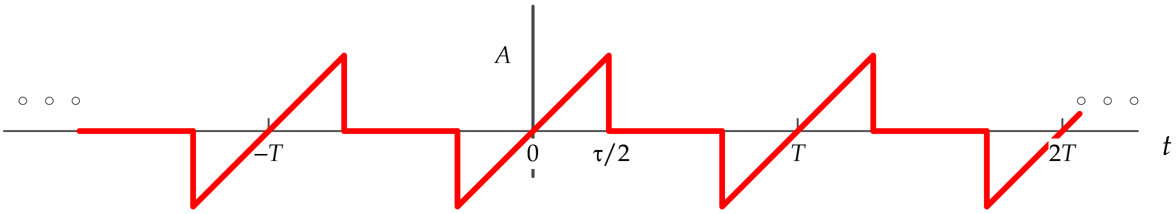
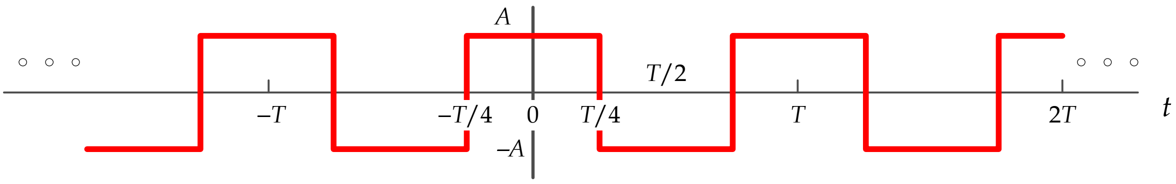


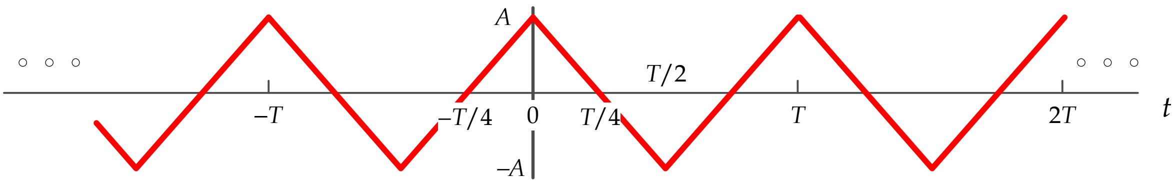

$rep_T\{A\ saw\ (t/\tau)\}$




$rep_T\{2A\ rect\ (2t/T)\} - A$




$rep_T\{2A\ \Delta\ (2t/T)\} - A$




$rep_T\{A\ saw\ (t/T)\}$