Lecture 15: Network of Queues

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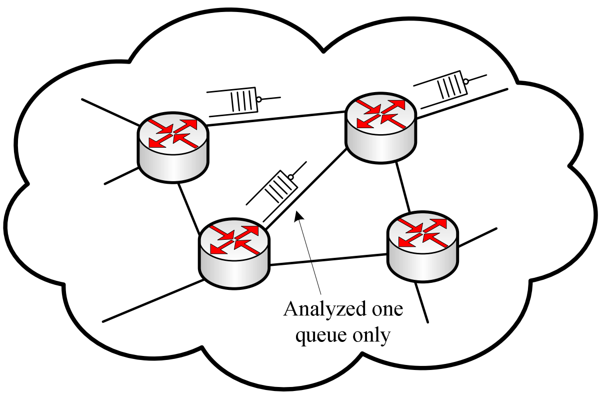
Network-Wide Performance

Data Network: multiple switches or routers



Network of Queues

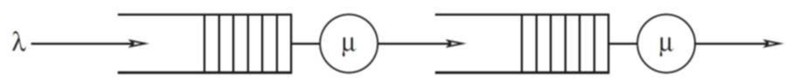
- A multiple-node system is one in which a customer requires service at more than one node.
- Each node is a service center having a queue and one (or more) servers to handle customer requests.
- Customers enter the system by arriving at one of the service centers, queue for and eventually receive service at this center.
- Upon departure from this service center, customers either proceed to another service center in the network to receive additional service, or else leave the network completely.

Assumptions

- Customers departing from one service center may mix with customers leaving a second service center and the combined flow may be destined to enter a third service center.
- We assume that the server at one service center is operating independently of the server at other service centers, though the service time of one customer might be similar (e.g. a packet of a certain size traversing different routers).
- In an open queuing network, customers enter and depart from the network. In a closed queuing network, customers neither enter nor depart from the network.

Difficulty

- Consider two queues in tandem, both having equal service rate μ.
- Arrival of packets to the first node is Poisson with rate λ pkts/sec.
- If all packets have equal length with transmission time $1/\mu$, then the first node is an M/D/1 queue (which can be analyzed using P-K equations).
- As long as the first node is not empty, packets will arrive at the second node deterministically every $1/\mu$ seconds (which is similar to D/D/1 system).
- If the first node is empty, a packet will not arrive at the second node until one arrives at the first node and then spends $1/\mu$ seconds in the first node.
- Hence, arrivals to the second node are no longer Poisson.



Burke's Theorem

- So, we limit ourselves to simpler systems that can give us some guarantees.
- Burke's Theorem states that for a single M/M/1, M/M/S, or M/M/∞ queueing system (i.e., with Poisson arrivals with arrival rate λ and exponential service time), and under steady state conditions, the following hold true for that single system:

1. The departure process from that system is also Poisson and with rate λ (which is throughput).
2. At each time t, the number of customers in the system is independent of the sequence of departure times prior to t.

Burke's Theorem [2]

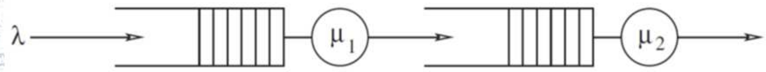
- The second part of Burke's theorem says that we cannot make any statements about the number of customers at a node after having observed a sequence of departures.
- For example, a flood of closely spaced departures does not imply that the last of these leaves behind a large number of customers, nor does the end of an observation period having few departures imply that these departures leave behind a largely empty system. Burke's theorem does not allow us to draw these conclusions.

Burke's Theorem [3]

- Hence, from second part of Burke's theorem, it follows that the number of customers present in node 1 is independent of the sequence of earlier arrivals at node 2. Consequently, it must also be independent of the number of customers present in node 2.
- It has been shown that the above systems, (i.e., M/M/1, M/M/S, or M/M/∞) are the only FCFS queues with this property. In other words, an M/G/1 and G/G/1 systems do not have this property in general.

Example

- Two M/M/1 queues in tandem.
- As a result of Burke's theorem, the distribution of customers in the two nodes is the same as if they were two isolated *independent* M/M/1 queues.
- Each has Poisson arrivals with rate λ.
- The service rate is different, though. It is μ_1 for the first queue and μ_2 for the second queue.
- For stability we require $\mu_1 > \lambda$ and $\mu_2 > \lambda$.

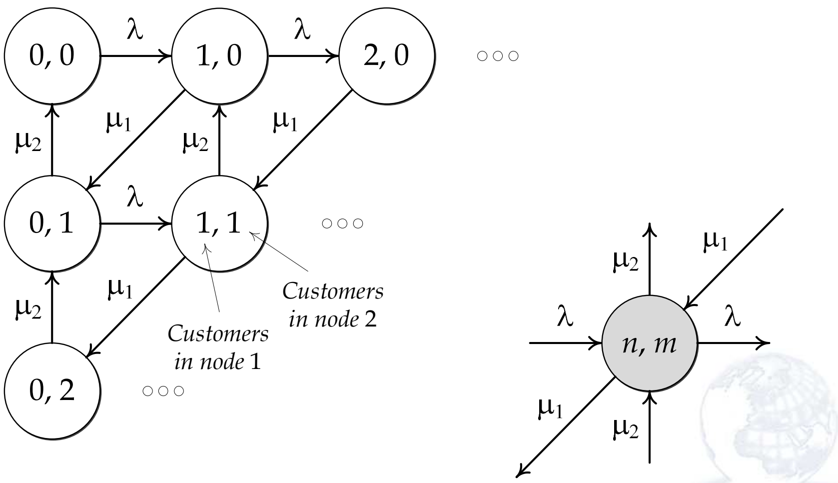


Use Burke's Theorem

- From second part of Burke's theorem, the number of customers present in node 1 is independent of the number of customers present in node 2.
- Therefore node 2 behaves like an M/M/1 queue, and can be analyzed independently of node 1.
- Hence,

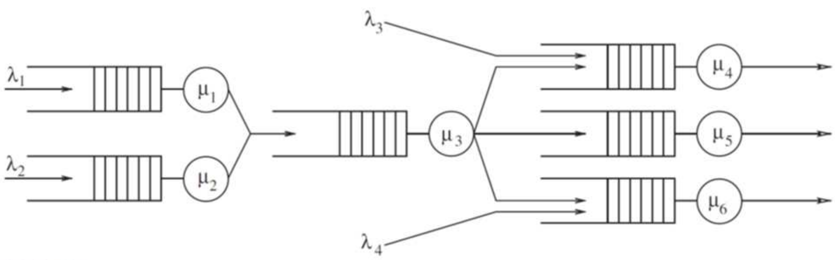
- Prob{n at node 1 and m at node 2} = Prob{n at node 1} × Prob{m at node 2}
- $\pi_{n,m} = \pi_n @node1 \times \pi_m @node2$
- $\pi_{n,m} = \rho_1^n(1-\rho_1) \times \rho_2^m(1-\rho_2)$

Just like two-dimensional CTMC



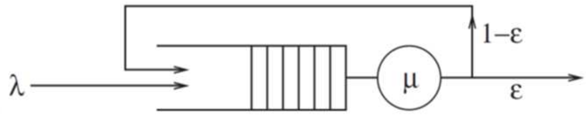
Open Networks: Feed forward

- In an open feed forward queuing network, a job *cannot* appear in the same queue for more than one time.
- Remember that aggregation of mutually independent Poisson processes is a Poisson process.



Open Networks: Feedback

- In an open feedback queuing network, after a job is served by a queue, it may reenter the same queue.
- The work of Jackson shows that, even in the presence of feedback loops, the individual nodes behave as if they were fed by Poisson arrivals.



Jackson Networks

- Consider an open network consisting of M nodes, i.e., there is at least one node to which customers arrive from an external source and there is at least one node from which customers depart from the system.
- The system is a Jackson network if the following are true for $i, j = 1, 2, \dots, M$:

1. Node i consists of an infinite FCFS queue and S_i exponential servers, each with parameter μ_i.
2. External arrivals to node i are Poisson with rate γ_i.
3. After completing service at node i, a customer will proceed to node j with probability r_{ij} independent of past history (notice that this permits the case where $r_{ii} \geq 0$) or will depart from the system, never to return again, with probability $1 - \sum_{j=1}^{M} r_{ij}$.

Routing Matrix & Traffic Equations

- The routing matrix $R = [r_{ij}]$ determines the permissible transitions between nodes.
- From this matrix we can determine the total average arrival rate of customers to each node i, denoted by λ_i, in steady-state (equilibrium):

$$\lambda_i = \gamma_i + \sum_{j=1}^{M} \lambda_j r_{ji}\,, \qquad i = 1,2,\ldots,M$$

- These are called the traffic equations for the network, and need to be solved to find λ_i.

Total Traffic Equation

- In steady state we have a traffic equation for the whole network:

$$\sum_{i=1}^{M} \gamma_i = \sum_{i=1}^{M} \lambda_j \left(1 - \sum_{j=1}^{M} r_{ji}\right)$$

$$\sum_{i=1}^{M} \gamma_i = \gamma = \sum_{i=1}^{M} \lambda_j P[leaving\ network\ from\ j]$$

- The above says that the total rate at which customers arrive into the queueing network from outside, γ, is equal to the rate at which they leave the network.
- Traffic equations represent a nonhomogeneous system of linear equations (nonzero right-hand side) with non-singular coefficient matrix. Its unique solution may be found using standard methods such as Gaussian elimination or matrix inversion.

Jackson Theorem

- For a Jackson network with effective arrival rate λ_i to node i, and assuming that $\lambda_i < S_i \mu_i$ for all i, the following are true in steady state:

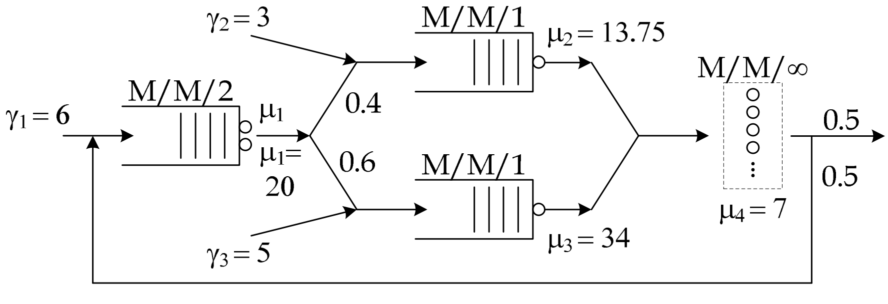
1. Node i behaves stochastically as if it were subjected to Poisson arrivals with rate λ_i.
2. The number of customers at any node is independent of the number of customers at every other node.

- Hence, for a Jackson network with M nodes:

$$\pi_{n_1, n_2, \ldots, n_M} = \pi_{n_1}@1 \times \pi_{n_2}@2 \times \cdots \times \pi_{n_M}@M$$

- This is known as the *product-form solution*.

Example



- An open network with Poisson arrivals with rate γ_i
- Example: Determine $\pi_{0,0,0,4}$
- Homework: Determine $\pi_{0,1,1,1} = 0.0009768$

Performance Parameters for Jackson Network

- Mean number of customers in the whole network:
$$\bar{n} = \overline{n_1} + \overline{n_2} + \cdots + \overline{n_M}$$
- Mean time spent in the network (use Little's Law):
$$\bar{\tau} = \frac{\bar{n}}{\gamma} = \frac{\overline{n_1} + \overline{n_2} + \cdots + \overline{n_M}}{\sum_{i=1}^{M} \gamma_i}$$
- Notice that $\bar{\tau} \neq \overline{\tau_1} + \overline{\tau_2} + \cdots + \overline{\tau_M}$
- Throughput at each node (infinite buffers):
$$\overline{Y_i} = \lambda_i$$
- Network throughput (infinite buffers):
$$\overline{Y} = \gamma = \sum_{i=1}^{M} \gamma_i$$

